ASYMPTOTIC PROPERTIES OF SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS WITH PERIODIC COEFFICIENTS IN LINEAR TERMS

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SUMMARY

Motivation: Many biological processes are modeled by delay differential equations. Therefore, for more deep insight into these processes, it is necessary to develop effective theoretical and numerical methods to study qualitative properties of solutions of such equations.

Results: In the present paper we give results concerning asymptotic stability of solutions of quasilinear systems of delay differential equations with periodic coefficients in linear terms. From these results we obtain conditions for asymptotic stability of stationary solutions of equations modeling substance synthesis and the blood cell production.

INTRODUCTION

Various biological processes are modeled by delay differential equations. For example, such equations arise when modeling of gene networks, population dynamics, the blood cell production, and etc. Therefore development of methods for study of qualitative properties of solutions of such equations has theoretical as well as practical importance.

The present paper is devoted to delay differential equations with periodic coefficients in linear terms

\[ \frac{d}{dt} y(t) = A(t)y(t) + B(t)y(t-\tau) + F(y(t), y(t-\tau)), \quad t > \tau > 0, \]  

where \( A(t), B(t) \) are \( n \times n \) matrices with continuous periodic entries, i.e.

\[ A(t + T) = A(t), \quad B(t + T) = B(t), \quad T > \tau, \]

\( F(u, v) \) is a real-valued vector-function satisfying the Lipschitz condition and

\[ \|F(u, v)\| \leq q_1 \|u\|^{\omega_1} + q_2 \|v\|^{\omega_2}, \quad q_1, q_2, \omega_1, \omega_2 \geq 0. \]

Our goal is to study asymptotic stability of the zero solution of the system (1) and obtain estimates characterizing decay rate as \( t \to \infty \). These results generalize the authors’ results from the paper (Demidenko, Matveeva, 2005).
In (Demidenko, Matveeva, 2005) a modification of the Lyapunov-Krasovskii functional was proposed. Using the modification, we obtained estimates of exponential decrease at infinity for solutions of linear and quasilinear systems of delay differential equations of the form (1) with constant coefficients. Attraction domains of the zero solution were established too.

Using the authors’ results from (Demidenko, Matveeva, 2001), in the present paper we obtain analogous results for linear and quasilinear systems of delay differential equations of the form (1) in the case of periodic coefficients.

**RESULTS**

At first, we consider the following linear system of delay differential equations with $T$-periodic coefficients

$$
\frac{d}{dt} y(t) = A(t)y(t) + B(t)y(t - \tau), \quad t > \tau.
$$

In the next theorem we give sufficient conditions of asymptotic stability of the zero solution of the system and establish estimates for solutions of (2).

**Theorem 1.** Suppose that there exist matrices

$$H(t) = H^*(t) \in C'[0,T] \quad \text{and} \quad K(s) = K^*(s) \in C'[0,\tau]$$

such that

$$H(0) = H(T) > 0, \quad K(s) > 0, \quad \frac{d}{ds} K(s) < 0, \quad s \in [0,\tau],$$

and the matrix

$$C(t) = -\left( \begin{array}{ccc} \frac{d}{dt} H(t) + H(t)A(t) + A^*(t)H(t) + K(0) & H(t)B(t) \\ B^*(t)H(t) & -K(\tau) \end{array} \right)$$

is positive definite on the interval $[0,T]$. Denote by $c_1(t) > 0$ the minimal eigenvalue of the matrix $C(t)$ and by $k > 0$ the maximal number such that

$$\frac{d}{ds} K(s) + kK(s) \leq 0, \quad s \in [0,\tau].$$

Then the zero solution of the system (2) is asymptotically stable; moreover, for a solution of (2) with an initial function $\varphi(t) \in C[0,\tau]$, the following inequality holds

$$\|y(t)\|^2 \leq h_1^{-1}(t) \exp \left( -\int_{\tau}^{t} \frac{\varepsilon(\xi)}{\|H(\xi)\|} \, d\xi \right) \left[ \langle H(\tau)\varphi(\tau), \varphi(\tau) \rangle + \int_{0}^{\tau} \left( K(\tau-s)\varphi(s), \varphi(s) \right) ds \right],$$

for $t > \tau$, where $h_1(t) > 0$ is the minimal eigenvalue of the matrix $H(t)$,

$$\varepsilon(t) = \min \{ c_1(t), kH(t) \}. \quad (3)$$

We now consider the quasilinear system (1). For simplicity, we formulate our result in the case $q_2 = 0$. 

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Theorem 2. Let the conditions of Theorem 1 be satisfied and

\[
r_{m/2} = \left(1 - \exp\left(-\frac{\omega_1 T}{2} \frac{e(s + \tau)}{\|H(s + \tau)\|} ds\right)\right) \times
\]

\[
\times \left[\frac{\omega_1 T}{2} \frac{e(s + \tau)}{\|H(s + \tau)\|} \exp\left(-\frac{\omega_1 T}{2} \frac{e(s + \tau)}{\|H(s + \tau)\|} ds\right) d\xi\right]^{-1},
\]

where \( e(t) > 0 \) is given by (3), \( h_1 > 0 \) is the minimal eigenvalue of the matrix \( H(t) \).

Then the zero solution of the system (1) is asymptotically stable, and the set of real-valued functions

\[
\Sigma = \left\{ \varphi(t) \in C[0, \tau] : \langle H(\tau)\varphi(\tau), \varphi(\tau) \rangle + \int_0^\tau \langle K(\tau - s)\varphi(s), \varphi(s) \rangle ds < r \right\}
\]

is an attraction domain of the zero solution. Moreover, for a solution of the system (1) with an initial function \( \varphi(t) \in \Sigma \), the following estimate holds

\[
\|y(t)\| \leq h_1^{-1}(t) \exp\left[-\int_0^\tau \frac{e(\xi)}{\|H(\xi)\|} d\xi\right]\left[\langle H(\tau)\varphi(\tau), \varphi(\tau) \rangle + \int_0^\tau \langle K(\tau - s)\varphi(s), \varphi(s) \rangle ds \right] \times
\]

\[
\times \left[1 - r_{m/2} \left[\langle H(\tau)\varphi(\tau), \varphi(\tau) \rangle + \int_0^\tau \langle K(\tau - s)\varphi(s), \varphi(s) \rangle ds \right] \right]^{-m/2}, \quad t > \tau.
\]

DISCUSSION

It is well-known that, for systems of linear delay differential equations with constant coefficients \( (A(t) = A, B(t) = B, F(u, v) = 0) \), there is an analog of the spectral theorem on asymptotic stability. Namely, if all roots of the quasipolynomial

\[
\det(A + e^{-\lambda\tau}B - \lambda I) = 0 \quad \text{belong to the left half-plane } C_\kappa = \{\lambda \in C : \text{Re}\lambda < 0\},
\]

then the zero solution is asymptotically stable. However, it is very hard to verify the condition. In the case of periodic coefficients an analogous spectral problem becomes practically unsolvable.

The results obtained in this paper make it possible to study qualitative properties of solutions of delay differential equations without finding roots of quasipolynomials. It provides ample opportunities to conduct computational investigations and obtain qualitative characteristics for biological processes.

We illustrate one of our results by the delay differential equation encountered in various biological problems (for modeling of gene networks (Likhoshvai et al., 2004), for describing the blood cell production (Kuang, 1993)).

\[
\frac{d}{dt} y(t) = -\theta y(t) + \frac{\alpha\beta^\gamma}{\beta^\gamma + y^\gamma(t - \tau)}, \quad t > \tau,
\]

where \( \alpha, \beta, \theta > 0, \gamma > 0 \) is integer. Consider a positive stationary solution \( y_\gamma \) of the equation (4). It is unique and defined by the equality

\[
\theta y_\gamma = \frac{\alpha\beta^\gamma}{\beta^\gamma + y_\gamma^\gamma}.
\]
From our results it follows that the stationary solution is asymptotically stable for $\theta > \frac{\alpha \gamma}{\beta}$.

Development of new methods for analysis of models describing dynamics of biological processes is one of actual problems of systemic biology. Many biological systems are modeled by means of delay differential equations. Examples of such models can be found in many books (see, for example, Murray, 1977; Marchuk, 1983; Kuang, 1993). In future, we plan to extend our results to various models.

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