MATRIX PROCESS MODELLING: ON PROPERTIES OF SOLUTIONS OF ONE DELAY DIFFERENTIAL EQUATIONS

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SUMMARY

Motivation: Many biological processes are modeled by systems of ordinary differential equations; moreover, orders of the systems may be very high. For example, synthesis of RNAs and proteins in gene networks involves hundreds and even thousands of intermediate stages. In the case of a sufficiently large number of intermediate stages, for a system modeling multi-stage substance synthesis without branching (Likhoshvai et al., 2004), we showed that this system can be replaced by one delay differential equation. However, we considered the zero initial conditions for the system. Obviously, initial conditions can be arbitrary in real problems.

Results: In the present paper we obtain a generalization of the limit theorem proved in (Likhoshvai et al., 2004) for the zero initial conditions in the case of arbitrary initial conditions.

INTRODUCTION

Modeling gene networks, it is necessary to take into consideration synthesis of tens and hundreds of thousands of intermediate stages of substances (DNA, RNA, proteins). Therefore, studying corresponding models, a researcher confronts with the high dimensionality problem.

In the paper (Likhoshvai et al., 2004) the high dimensionality problem was studied for a system of ordinary differential equations modeling substance synthesis without branching

\[
\begin{align*}
\frac{dx_i}{dt} &= g(x_n) - \frac{n - 1}{\tau} x_i, \\
\frac{dx_i}{dt} &= \frac{n - 1}{\tau} (x_{i-1} - x_i), \quad i = 2, ..., n - 1, \\
\frac{dx_n}{dt} &= \frac{n - 1}{\tau} x_{n-1} - \theta x_n, \quad \tau > 0, \quad \theta \geq 0.
\end{align*}
\]

In the case \( n \gg 1 \), in the paper (Likhoshvai et al., 2004) a new method for finding an approximate solution of the Cauchy problem with the zero initial conditions \( x_{n,0} = 0 \)
was proposed. The main idea of this method is to enlarge unrestrictedly the system (1) and to study the limit of the sequence consisting of the last components of solutions of the Cauchy problem. As was proved (Likhoshvai et al., 2004), for \( g(z) \in C^i(R) \) we have a uniform convergence \( x_n(t) \to y(t), \ n \to \infty, \ t \in [0, T] \), where \( y(t) \) is a solution of the following delay differential equation

\[
\frac{d}{dt} y(t) = -\theta y(t) + g(y(t - \tau)), \quad t > \tau; \tag{2}
\]

moreover,

\[
y(t) = 0, \quad t \in [0, \tau]. \tag{3}
\]

Hence, to find approximately the last component \( x_n(t) \) for \( n >> 1 \) it is enough to solve the initial problem (2), (3). Then we obtain \( x_n(t) \approx y(t) \).

In the present paper we continue to study connections between solutions of the system (1) for \( n >> 1 \) and solutions of the delay differential equation (2) when initial conditions are arbitrary (Likhoshvai et al., 2004; Demidenko et al., 2006).

**RESULTS**

Consider the Cauchy problem for the system (1) with initial conditions

\[
x \big|_{t=0} = x_0. \tag{4}
\]

Suppose that the function \( g(z) \in C(R) \) is bounded and satisfies the Lipschitz condition. Then the problem (1), (4) has a unique solution \( x(t) = (x_1(t), ..., x_n(t))^T \) on any interval \( [0, T] \). Let us increase unrestrictedly the number of equations in (1) and consider only the last components \( x_n(t) \) of solutions of the Cauchy problems. Then we obtain a sequence \( \{x_n(t)\} \) on the interval \( [0, T] \).

**Theorem 1.** Let the initial conditions in the Cauchy problem (1), (4) have the form \( x_0 = (a_1, ..., a_k, 0, ..., 0)^T \), where \( k \) does not depend on \( n \). Then:

a) for any \( T > \tau \) the convergence holds

\[
\|x_n(t) - y(t), L_p(0,T)\| \to 0, \quad p \geq 1, \quad n \to \infty; \tag{5}
\]

b) the limit function \( y(t) \in W^1_p(\tau, T) \) is a weak solution of the initial problem for the delay differential equation

\[
\frac{d}{dt} y(t) = -\theta y(t) + g(y(t - \tau)), \quad t > \tau, \tag{6}
\]

\[
y(t) = 0, \quad t \in [0, \tau), \quad y(\tau + 0) = \sum_{i=1}^{k} a_i; \quad
\]
c) the function \( y(t) \) is a classic solution of the equation (2) for \( t > 2\tau \).

Note that the condition indicated in Theorem 1 on the initial vector \( x_0 \) is essential. If we will consider initial vectors with finite number of nonzero components \( a_k \), where \( k \) depends on \( n \), then we can establish the convergence (5) to a weak solution of the delay differential equation (2). However, initial conditions will differ from the initial conditions in (6). We give a few examples below.

For simplicity, we assume that \( \theta = 0 \), \( \tau = 1 \). In the next theorems we consider a few examples of initial conditions in the Cauchy problem (1), (4). These examples show that, as the number of equations increases unrestrictedly, the last components \( x_n(t) \) tend to weak solutions of the equation (2) with initial conditions of the form

\[
y(t) = \varphi_{m,k}(t), \quad 0 \leq t \leq 1, \quad m \in N, \quad k = 1, 2, ..., 2^m.
\]

where

\[
2^{\frac{m}{2}}, \quad t \in [(k-1)/2^m, (k-1/2)/2^m), \\
\varphi_{m,k}(t) = -2^{\frac{m}{2}}, \quad t \in [(k-1/2)/2^m, k/2^m), \\
0, \quad t \notin [(k-1)/2^m, k/2^m).
\]

Note that the system of functions \( \{\varphi_{m,k}(t)\} \) forms the orthonormal Haar basis in the space \( L_2(0,1) \) (see, for instance, (Triebel, 1983)).

**Theorem 2.** Let \( n = 4l \). Assume that the initial vector in the Cauchy problem (1), (4) has components

\[
a_{n/2} = \sqrt{2}, \quad a_{3n/4} = -2\sqrt{2}, \quad a_n = \sqrt{2}, \quad \text{the rest of components} \quad a_k = 0,
\]

or

\[
a_{n/4} = -2\sqrt{2}, \quad a_{n/2} = \sqrt{2}, \quad \text{the rest of components} \quad a_k = 0.
\]

Then, for any \( T > \tau \), the convergence (5) holds and the limit function \( y(t) \) is a weak solution of the equation (2) with the initial conditions \( \varphi_{1,1}(t) \) or \( \varphi_{1,2}(t) \), respectively.

**Theorem 3.** Let \( n = 8l \). Assume that the initial vector in the Cauchy problem (1), (4) has components

\[
a_{3n/4} = 2, \quad a_{7n/8} = -4, \quad a_n = 2, \quad \text{the rest of components} \quad a_k = 0,
\]

or

\[
a_{n/2} = 2, \quad a_{5n/8} = -4, \quad a_{3n/4} = 2, \quad \text{the rest of components} \quad a_k = 0,
\]

or

\[
a_{n/4} = 2, \quad a_{3n/8} = -4, \quad a_{n/2} = 2, \quad \text{the rest of components} \quad a_k = 0,
\]

or

\[
a_{n/8} = -4, \quad a_{n/4} = 2, \quad \text{the rest of components} \quad a_k = 0.
\]

Then, for any \( T > \tau \), the convergence (5) holds and the limit function \( y(t) \) is a weak solution of the equation (2) with the initial conditions \( \varphi_{2,1}(t) \) or \( \varphi_{2,2}(t) \) or \( \varphi_{2,3}(t) \) or \( \varphi_{2,4}(t) \), respectively.
Analogous assertions can be formulated for the Cauchy problem for \( n = 2^{m+1} l \), \( \theta = 0 \), \( \tau = 1 \). Thus, one can establish that \( x_n(t) \approx y(t) \) for \( n >> 1 \), and \( y(t) \) is a weak solution of the equation (2) with initial conditions of the form (7).

**DISCUSSION**

Using analogs of the formulated theorems, we can point out vectors \( x_0 \) of initial conditions in (4) under which the last components of solutions of the Cauchy problems of the form (1), (4) for \( n >> 1 \) approximate solutions of delay differential equations of the form (2) with initial conditions of the form

\[
y(t) = \varphi(t), \quad t \in [0,1], \quad \varphi(t) \in C[0,1].
\]

(9)

To this end, it is sufficient to write an expansion of the function \( \varphi(t) \) in a series in the Haar functions (8) in \( L^2(0,1) \)

\[
\varphi(t) = \sum_{m=1}^{\infty} \sum_{k=1}^{2^m} c_{m,k} \varphi_{m,k}(t),
\]

(10)

where

\[
c_{m,k} = \int_0^1 \varphi_{m,k}(s) \varphi(s) ds.
\]

(11)

For any \( \varepsilon > 0 \) we can find a number \( m_\varepsilon \) such that

\[
\left\| \varphi(t) - \sum_{m=1}^{m_\varepsilon} \sum_{k=1}^{2^m} c_{m,k} \varphi_{m,k}(t), L^2(0,1) \right\| \leq \varepsilon / 2.
\]

Consider the Cauchy problem (1), (4) for \( n = 2^{m+1} l \). Using analogs of Theorem 1–3, taking into account (10), (11), we choose a relevant initial vector \( x_0 \). Then, in the same way as in (Demidenko et al., 2006), for any \( \varepsilon > 0 \) there exists \( n_\varepsilon \) such that, for any \( n > n_\varepsilon \), the last component of a solution of the Cauchy problem (1), (4) satisfies the inequality

\[
|x_n(t) - y(t)| \leq \varepsilon, \quad t \in [\tau, T],
\]

where \( y(t) \) is a solution of the initial problem (2), (9).

Our result makes it possible to solve the high dimensionality problem for the system of ordinary differential equations (1) modeling substance synthesis without branching in the case of arbitrary initial conditions. We plan to extend the result to other systems modeling gene networks.
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