MATRIX PROCESS MODELLING:
ON A NEW METHOD OF APPROXIMATION
OF SOLUTIONS OF DELAY DIFFERENTIAL
EQUATIONS

Demidenko G.V.∗1, Mudrov A.V.2
1 Sobolev Institute of Mathematics, SB RAS, Novosibirsk, 630090, Russia; 2 Novosibirsk State
University, Novosibirsk, 630090, Russia
∗ Corresponding author: e-mail: demidenk@math.nsc.ru

Key words: gene networks, mathematical models, ordinary differential equations, delay differential
equations, initial problem

SUMMARY

Motivation: Matrix processes of replication, transcription, and translation, including
hundreds and thousands of elongation stages of the same type, are essential part of natural
and artificial genetic systems. To reflect adequately these processes in models of gene
networks it is necessary to elaborate effective theoretical and numerical mathematical
methods.

Results: In the present paper we solve the problem on approximation of solutions of
delay differential equations by solutions of systems of ordinary differential equations.
This result gives a new method in order to study biological processes described by means
of systems of ordinary differential equations of very high orders; for example, multi-stage
substance synthesis.

INTRODUCTION

In the present paper we continue to study connections between solutions of systems of
a large number of ordinary differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= qmy_n - \frac{n-1}{\tau} x_1, \\
\frac{dx_i}{dt} &= \frac{n-1}{\tau} (x_{i-1} - x_i), & i = 2, \ldots, n-1, \\
\frac{dx_n}{dt} &= \frac{n-1}{\tau} x_{n-1} - mx_n, \\
\frac{dy_n}{dt} &= f(y_n, x_n), & 1 > q > 0, \quad \tau > 0, \quad m \in N,
\end{align*}
\]

(1)

and solutions of the delay differential equation

\[
\frac{dy(t)}{dt} = f(y(t), qy(t-\tau)), \quad t > \tau.
\]

(2)
We assume that the function \( f(u,v) \) is bounded and satisfies the Lipschitz condition:

\[
\sup_{u,v \in R} |f(u,v) - f(u_1,v_1)| \leq L_1 |u - u_1| + L_2 |v - v_1|.
\]

Let \( y(t) \) be a solution of the equation (2), satisfying an initial condition

\[
y(t) = \varphi(t), \quad 0 \leq t \leq \tau,
\]

where the function \( \varphi(t) \) is a solution of the problem

\[
\frac{d}{dt} \varphi(t) = f(\varphi(t),0), \quad \varphi|_{t=0} = y_0.
\]

It was established (Demidenko, Likhoshvai, 2005) that \( y(t) \) can be represented as the repeated limit

\[
\lim_{m \to \infty} \lim_{n \to \infty} y_n^m(t) = y(t),
\]

where \( y_n^m(t) \) is the last component of a solution of the Cauchy problem for the system (1) with the initial conditions

\[
x_1|_{t=0} = \ldots = x_n|_{t=0} = 0, \quad y_n|_{t=0} = y_0.
\]

Note that the representation (5) was proved in (Demidenko, Likhoshvai, 2005) on the interval \([0,t_0]\), where

\[
\tau < t_0 < \min \left\{ \frac{1-q}{L_1}, \frac{1}{L_2} \right\}.
\]

Consequently, on “small” interval \([0,t_0]\) solutions of initial problems of the form (2), (3) for delay differential equations can be approximated by solutions of the Cauchy problems of the form (1), (6) for systems of a large number of ordinary differential equations:

\[
y(t) \approx y_n^m(t), \quad m >> 1, \quad n \gg 1.
\]

In the present paper we strengthen our result (Demidenko, Likhoshvai, 2005) and prove that (5) holds on any interval \([0,T]\). We establish also uniform estimates for (7) and point out rate of the convergence (5).

RESULTS

We formulate the main results of the paper below.

Consider a series of the Cauchy problems of the form (1), (6) with a fixed \( m \) by unbounded increasing the number of equations \( n \). Solving each problem and considering
only the last two components of solutions, we obtain a sequence of vector-functions 
\[ \{z_n^m (t)\}, \quad z_n^m (t) = (x_n^m (t), y_n^m (t)) . \]

**Theorem 1.** The sequence \( \{z_n^m (t)\} \) converges uniformly as \( n \to \infty \) on the interval \([0, T]\)
\[ x_n^m (t) \to x^m (t), \quad y_n^m (t) \to y^m (t) ; \]
moreover, the limit vector-function \( z^m (t) = (x^m (t), y^m (t)) \) is a solution of the system of integral equations
\[ x^m (t) = qm \int_0^{t-\tau} e^{-m(t-s)} y^m (s) ds , \quad t > \tau , \quad (8) \]
\[ y^m (t) = y_0 + \int_0^t f(y^m(s), x^m(s)) ds , \quad (9) \]
and \( x^m (t) = 0 , \quad t \in [0, \tau] . \)

**Theorem 2.** The system of the integral equations (8), (9) has a unique solution \( z^m (t) = (x^m (t), y^m (t)) \) continuous on the interval \([0, T]\), and \( x^m (t) = 0 \) for \( t \in [0, \tau] \).

Obviously, the functions (8), (9) satisfy the system of the delay differential equations
\[ \frac{d}{dt} x^m (t) = -mx^m (t) + qmy^m (t - \tau) , \quad \frac{d}{dt} y^m (t) = f(y^m (t), x^m (t)) . \quad (10) \]

Consider a sequence of the integral equations (8), (9) by unbounded increasing \( m \).
Solving each system of the form (8), (9), we obtain the sequence \( \{z^m (t)\} \) of the vector-function \( z^m (t) = (x^m (t), y^m (t)) \).

**Theorem 3.** The sequence \( \{z^m (t)\} \) is convergent on the interval \([\tau, T]\):
\[ x^m (t) \to x(t) , \quad y^m (t) \to y(t) , \quad m \to \infty ; \]
moreover,
\[ x(t) = qy(t - \tau) , \]
\[ y(t) = y_0 + \int_0^t f(y(s), 0) ds , \quad t \in [0, \tau] , \]
\[ y(t) = y_0 + \int_0^\tau f(y(s), 0) ds + \int_\tau^t f(y(s), qy(s - \tau)) ds , \quad t \in [\tau, T] . \]
From Theorem 3 we have
\[ y(t) \in C[0, T] \cap C^1(0, \tau) \cap C^1(\tau, T); \]
moreover, \( y(t) \) is a solution of the problem (2), (3) for the delay differential equation with the initial function (4).

The next theorems give estimates for rate of the convergence (5).

**Theorem 4.** For any fixed \( m \in \mathbb{N} \), the limit relations hold as \( n \to \infty \)
\[
\max_{t \in [0,T]} \left| x_n^m(t) - x_n^m(t) \right| = O\left(n^{-1/4}\right),
\]
\[
\max_{t \in [0,T]} \left| y_n^m(t) - y_n^m(t) \right| = O\left(n^{-1/4}\right).
\]

**Theorem 5.** The asymptotic equalities hold as \( m \to \infty \)
\[
\max_{t \in [j\tau, (j+1)\tau]} \left| \alpha(t) - \alpha^m(t) \right| = O\left(\frac{\ln m}{n}\right), \quad \delta \in (0, \tau),
\]
\[
\max_{t \in [j\tau, (j+1)\tau]} \left| y(t) - y^m(t) \right| = O\left(\frac{\ln m}{n}\right)
\]
where \([j\tau, (j+1)\tau] \subset [0, T]\).

**DISCUSSION**

Form Theorem 1–5 we obtain a method for approximation of solutions of delay differential equations of the form (2) with the conditions (3), (4) by using solutions of the Cauchy problems of the form (1), (6).

On the other hand, approximate finding the last two components of solutions of (1) with a sufficiently large number of differential equations and the initial conditions (6) can be obtained by means of solutions of the system of the integral equations (8), (9) or by using a corresponding initial problem for the system of two delay differential equations (10). In the case of sufficiently large \( m \), approximate constructing the last component of a solution of (1) is reduced to solving the initial problem (2)–(4).

These results give a new method in order to study biological processes described by means of systems of ordinary differential equations of very high orders. Indeed, systems of high orders can be replaced by one or two delay differential equations. Thus, an analogous result was obtained in (Likhoshvai et al., 2004) for a system of differential equations modeling multi-stage substance synthesis without branching. As is known, synthesis of RNAs and proteins in gene networks involves a sufficiently large number (hundreds or even thousands) of intermediate stages. In this case, our results show that if the rate of each of the intermediate stages is sufficiently high, the kinetics of the end product output is practically independent of the kinetics if the internal synthesis stages. Everything depends on the regulatory mechanism of starting first stage of the synthesis and the delay time, which equals the average total time of all intermediate stages.
ACKNOWLEDGEMENTS

The work was supported by the Russian Foundation for Basic Research (Nos 04-01-00458, 05-04-49068, 05-07-90274), the Siberian Branch of the Russian Academy of Sciences (Interdisciplinary integration project No. 24). The authors are grateful to Vitaly Likhoshvai for valuable discussions.

REFERENCES
