MATRIX PROCESS MODELLING: 
ON ONE CLASS OF INFINITE-ORDER SYSTEMS 
OF DIFFERENTIAL EQUATIONS 
AND ON DELAY DIFFERENTIAL EQUATIONS 

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SUMMARY 

Motivation: In recent years, intensive study of gene networks, genetically controlled metabolic paths, signal transduction paths, and other complex genetic-molecular systems has been started. Presently, these studies are reaching a qualitatively new level due to wide use of microarray analysis, which makes it possible to reveal functions of many hundreds and even thousands of genes in a single experiment. Analysis of huge amounts of experimental data, which reflect complex processes in genetic-molecular systems, is impossible without efficient mathematical methods. 

Results: In the present paper we consider a class of models describing matrix branching processes with an unrestrictedly increasing number of stages. We show that the last components of solutions of corresponding systems of ordinary differential equations tend to a solution of one delay differential equation. 

INTRODUCTION 

Study of gene networks is one of the main problems of systemic biology. For this purpose methods of numerical and theoretical mathematical modeling are extensively used. Modeling makes it possible to take into account synthesis of tens and hundreds of thousands of intermediate states of DNA, RNA, and proteins. However, from a mathematical standpoint, the consideration of intermediate stages of synthesis leads to systems of ordinary differential equations of a very high order. Thus, the high dimensionality problem arises when modeling gene networks. 

As a rule, solving systems with a large number of differential equations, researchers try to reduce the problem to solving systems of substantially smaller orders. At the present time, there is a huge number of works devoted to various methods of reduction of orders. In the paper (Likhoshvai et al., 2004), studying a model of multi-stage substance synthesis without branching
\[
\frac{dx_i}{dt} = g(x_n) - \frac{x_i}{\tau}, \quad i = 1, \ldots, n
\]
\[
\frac{dx_i}{dt} = \frac{n-1}{\tau} (x_{i-1} - x_i), \quad i = 2, \ldots, n-1
\]
\[
\frac{dx_n}{dt} = \frac{n-1}{\tau} x_{n-1} - \theta x_n, \quad \tau > 0, \quad \theta \geq 0
\]

(1)

it was proposed and substantiated a new method to research systems of ordinary differential equations of sufficiently high orders. Keeping structures of systems, we proposed to increase unrestrictedly the number of equations of the systems and to consider solutions of limit systems.

Such an approach for the system of multi-stage substance synthesis without branching (1) for \( n \gg 1 \) allows us to find approximately the end product, i.e., the component \( x_n(t) \).

As was shown in (Likhoshvai et al., 2004), there is a close connection between solutions of the system (1) as \( n \to \infty \) and solutions of the delay differential equation

\[
\frac{dy(t)}{dt} = -\theta y(t) + g(y(t-\tau)), \quad t > \tau.
\]

(2)

In particular, if we consider the Cauchy problem for (1) with the zero initial conditions and \( g(x) \in C^1(R) \), then \( x_i(t) \to y(t), \quad n \to \infty, \quad t \in [0,T] \), where \( y(t) \) is a solution of (2); moreover, \( y(t) = 0, \quad t \in [0,\tau] \). Consequently, to find approximately the end product we need not solve the Cauchy problem for systems of the form (1) of high orders. It suffices to solve an initial problem for one delay differential equation. Then \( x_n(t) \approx y(t) \) as \( n \to \infty \); moreover, we can write estimates for the approximation (see Likhoshvai et al., 2004).

A development of this approach for constructing approximate solutions of ordinary differential equations of high orders can be found in the papers (Demidenko, Likhoshvai, 2005; Demidenko et al., 2006).

In the present paper we point out a class of systems of ordinary differential equations

\[
\frac{dx}{dt} = A_n x + F_n(t,x)
\]

(3)

whose approximate solutions can be constructed by the method mentioned above. In particular, we establish a connection between these solutions and solutions of delay differential equations of the form (2). This class of systems includes the system of multi-stage substance synthesis with nonconstant rates of transition from the \( i \) th stage to the \( i+1 \) st stage.

RESULTS

Consider a series of systems of differential equations of the form (3). Each of the systems consists of \( n \) ordinary differential equations, linear terms are defined by an \( n \times n \) numerical matrix \( A_n \), and nonlinear terms are given by a vector-function \( F_n(t,x) \) of the form

\[
F_n(t,x) = (g(t,x_n), 0, \ldots, 0)^T.
\]
We consider the series of systems of ordinary differential equations of the form (2) under the following conditions on the sequence \( \{A_n\} \) of matrices and the function \( g(t,z) \).

1. Let \( \lambda_1^n, \ldots, \lambda_n^n \) be the eigenvalues of the matrix \( A_n \). Suppose that

\[
\lambda_1^n \to -\theta, \quad n \to \infty, \quad \text{Re} \lambda_j^n \leq -\frac{n-1}{\tau} + \lambda_0, \quad j = 2, \ldots, n,
\]

where \( \lambda_0, \theta \geq 0, \tau > 0 \) are constant.

2. Assume that the cofactor \( \alpha_n \) of the element \( b_{1,n} \) of the matrix \( (\lambda I - A) = (b_{i,j}) \) does not depend on \( \lambda \), and

\[
|\alpha_n| \left( \frac{n-1}{\tau} \right)^{-n} \leq a < \infty.
\]

3. Suppose that the convergence holds

\[
\frac{1}{\alpha_n} \prod_{j=2}^{n} (\lambda_1^n - \lambda_j^n) \to e^{-\theta \tau}, \quad n \to \infty.
\]

4. Assume that the function \( g(t,z) \) is bounded and satisfies the Lipschitz condition

\[
\sup_{z \in \mathbb{R} \setminus 0} |g(t,z_1) - g(t,z_2)| \leq L|z_1 - z_2|, \quad z_1, z_2 \in \mathbb{R}.
\]

For each of the systems of the form (3) we consider the Cauchy problem with the zero initial conditions

\[
\begin{align*}
\frac{dx}{dt} &= A_n x + F_n(t,x), \\
x|_{t=0} &= 0.
\end{align*}
\]  

(4)

We will increase unrestrictedly the number of equations of the system and consider only the last component of a solution of the Cauchy problem (4). Then we obtain a sequence \( \{x_n(t)\} \) on the interval \([0, T]\).

**Theorem 1.** The sequence \( \{x_n(t)\} \) converges uniformly on the interval \([0, T]\), and the limit function \( y(t) \) is a solution of the initial problem for a delay differential equation

\[
\begin{align*}
\frac{dy}{dt} &= -\theta y(t) + g(t-\tau, y(t-\tau)), \quad t > \tau, \\
y(t) &= 0, \quad t \in [0, \tau].
\end{align*}
\]

We can show that an analog of Theorem 1 holds for a series of the Cauchy problem for systems of the form (3) with nonzero initial conditions. However, in this case we observe an interesting peculiarity that the sequence \( \{x_n(t)\} \) converges in the \( L_p(0,T) \) space, and the limit function \( y(t) \) is a weak solution of a delay differential equation. We formulate this result in the case when the vector of initial conditions in the Cauchy problem (4) has a nonzero first component, while the others vanish, i.e.,

\[
\begin{align*}
x|_{t=0} &= x_0, \quad x_0 = (a,0,\ldots,0)^T, \quad a \neq 0.
\end{align*}
\]

**Theorem 2.** For any \( T > \tau \) the convergence holds

\[
\left\| x_n(t) - y(t), L_p(0,T) \right\| \to 0, \quad p \geq 1, \quad n \to \infty;
\]
moreover, \( y(t) \) is a weak solution of the initial problem for a delay differential equation

\[
\frac{d}{dt} y(t) = -\theta y(t) + g(t-\tau, y(t-\tau)), \quad t > \tau, \\
y(t) = 0, \quad t \in [0, \tau), \quad y(\tau + 0) = a.
\]

Complete proofs of corresponding assertions can be found in (Demidenko et al., 2006).

**DISCUSSION**

From the proofs of the theorems in (Demidenko et al., 2006) we obtain uniform estimates of the difference module

\[
|x_n(t) - y(t)| \leq \beta(n), \quad t \in [\tau, T], \quad \beta(n) = O(n^{-1/4}), \quad n \to \infty.
\]  (5)

Consequently, we have an effective method for constructing an approximation of the \( n \)th component \( x_n(t) \) of the solution of the Cauchy problem (4) for \( n \gg 1 \). Namely, to find approximately \( x_n(t) \) we can solve the initial problem for the delay differential equation and use the estimate (5) in order to obtain accuracy of the approximation \( x_n(t) \approx y(t), \quad n \gg 1 \). Obviously, using this method, we can study qualitative properties of the \( n \)th component \( x_n(t) \).

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